International Research Journal of Education and Technology
Peer Reviewed Journal
ISSN 2581-7795

A Portrayal of Integer Solutions to Non-homogeneous Ternary Cubic Diophantine Equation

$$5(x^2+y^2)-6xy+4(x+y+1)=6100z^3$$

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Abstract

This paper aims at determining varieties of non-zero distinct integer solutions to non-homogeneous ternary cubic diophantine equation

$$5(x^2+y^2)-6xy+4(x+y+1)=6100z^3$$

Keywords : ternary cubic , non-homogeneous cubic ,integer solutions

Introduction

It is well-known that the Diophantine equations ,homogeneous or non-homogeneous ,have aroused the interest of many mathematicians. In particular ,one may refer [1-10] for cubic equations with three and four unknowns.

While collecting problems on third degree diophantine equations ,the problem of getting integer solutions to the non-homogeneous ternary cubic diophantine equation given by $5(x^2+y^2)-6xy+4(x+y+1)=6100z^3$ [11] has been noticed. The authors of [11] have presented three sets of integer solutions to the cubic equation considered in [11]. The main thrust of this paper is to exhibit other sets of integer solutions to ternary non-homogeneous cubic equation given by $5(x^2+y^2)-6xy+4(x+y+1)=6100z^3$ in [11] by using elementary algebraic methods. The outstanding results in this study of diophantine equation will be useful for all readers.

Method of analysis

The non-homogeneous ternary cubic diophantine equation to be solved is given by

$$5(x^2+y^2)-6xy+4(x+y+1)=6100z^3 \tag{1}$$

Introduction of the linear transformations

$$x=5(u+v)-1, y=5(u-v)-1 \tag{2}$$

in (1) leads to

$$u^2+4v^2=61z^3 \tag{3}$$

The process of obtaining different sets of integer solutions to (1) is illustrated below :

Illustration 1

It is observed that (3) is satisfied by

$$u=5\alpha^{3s}, v=3\alpha^{3s} \tag{4}$$

and

$$z=\alpha^{2s} \tag{5}$$

Using (4) in (2) , we get

$$x=40\alpha^{3s}-1, y=10\alpha^{3s}-1 \tag{6}$$

Thus , (5) & (6) represent the integer solutions to (1).

Illustration 2

Taking

$$u = k\,v \tag{7}$$

in (3) leads to

$$(k^2+4)v^2 = 61\,z^3$$

which is satisfied by

$$v = 61^2\,(k^2+4)\,s^{3\alpha} \tag{8}$$

and

$$z = 61\,(k^2+4)\,s^{2\alpha} \tag{9}$$

In view of (7) , note that

$$u = 61^2\,k\,(k^2+4)\,s^{3\alpha} \tag{10}$$

Using (8) & (10) in (2) , we get

$$x = 5*61^2\,(k+1)(k^2+4)s^{3\alpha} - 1,\; y = 5*61^2\,(k-1)(k^2+4)s^{3\alpha} - 1 \tag{11}$$

Thus , (9) & (11) represent the integer solutions to (1).

Illustration 3

Taking

$$v = k\,u \tag{12}$$

in (3) leads to

$$(4k^2+1)\,u^2 = 61\,z^3$$

which is satisfied by

$$u = 61^2\,(4k^2+1)\,s^{3\alpha} \tag{13}$$

and

$$z = 61\,(4k^2 + 1)\, s^{2\alpha} \tag{14}$$

In view of (12) , note that

$$v = 61^2\, k\, (4k^2 + 1)\, s^{3\alpha} \tag{15}$$

Using (13) & (15) in (2) , we get

$$x = 5*61^2\,(k+1)(4k^2+1)s^{3\alpha} - 1,\ y = 5*61^2\,(1-k)(4k^2+1)s^{3\alpha} - 1 \tag{16}$$

Thus , (14) & (16) represent the integer solutions to (1).

Illustration 4

Taking

$$v = z \tag{17}$$

in (3) , it is written as

$$u^2 = z^2\,(61z - 4) \tag{18}$$

It is possible to choose the values of z so that the R.H.S. of (18) is a perfect square and hence the corresponding values of u are obtained.

In view of (17) ,the values of v are found .Substituting these values of v, u in (2) , the respective integer solutions to (1) are found. The above process is exhibited below:

Let

$$\alpha^2 = 61z - 4 \tag{19}$$

which is satisfied by

$$z_0 = 8, \alpha_0 = 22$$

Assume

$$\alpha_1 = h - \alpha_0 , z_1 = z_0 + k h \qquad (20)$$

to be the second solution to (19). Substituting (20) in (19) and simplifying,

we have

$$h = 2\alpha_0 + 61k$$

In view of (20) , one has

$$\alpha_1 = \alpha_0 + 61k , z_1 = z_0 + k(2\alpha_0 + 61k)$$

The repetition of the above process leads to the general solution to (19) as

$$\alpha_n = \alpha_0 + 61kn = 22 + 61kn ,$$
$$z_n = 2nk\alpha_0 + 61k^2 n^2 + z_0 = 44kn + 61k^2 n^2 + 8 \qquad (21)$$

From (18) , it is seen that

$$u_n = (61kn + 22)\,(61n^2 k^2 + 44kn + 8)$$

Also , from (17) ,note that

$$v_n = (61n^2 k^2 + 44kn + 8)$$

In view of (2) ,the integer solutions to (1) are given by

$$x_n = 5(61kn + 23)(61k^2 n^2 + 44kn + 8) - 1,$$
$$y_n = 5(61kn + 21)(61k^2 n^2 + 44kn + 8) - 1$$

alongwith (21) .

Illustration 5

Taking

$$u = z = 4s + 1 \qquad (22)$$

in (3) , it is written as

$$v^2 = (4s+1)^2 (61s+15)$$

Following the process as in Illustration 4 ,the corresponding integer solutions to (1) are found to be

$$x_n = 5(61n+26)(244n^2+200n+41)-1,$$
$$y_n = -5(61n+24)(244n^2+200n+41)-1,$$
$$z_n = (244n^2+200n+41).$$

Illustration 6

Assume

$$z = a^2 + 4b^2 \tag{23}$$

Express the integer 61 on the R.H.S. of (3) as the product of complex Conjugates as follows

$$61 = (5+i6)(5-i6) \tag{24}$$

Substituting (23) & (24) in (3) and employing the method of factorization , consider

$$u + i2v = (5+i6)(a+i2b)^3 \tag{25}$$

Equating the real and imaginary parts in (25) , the values of u, v are found.

In view of (2) ,the values of x, y are given by

$$\begin{aligned} x &= 5[8(a^3-12ab^2)-7(3a^2b-4b^3)]-1, \\ y &= 5[2(a^3-12ab^2)-17(3a^2b-4b^3)]-1 \end{aligned} \tag{26}$$

Thus , (23) & (26) give the integer solutions to (1).

Note 1:

Apart from (24) , one may consider the integer 61 on the R.H.S. of (3) as

$$61=(6+i5)(6-i5)$$

giving a different set of integer solutions to (1).

Conclusion:

In this paper, we have made an attempt to find infinitely many non-zero distinct integer solutions to the non- homogeneous cubic equation with three unknowns given by $5\left(x^2+y^2\right)-6xy+4(x+y+1)=6100z^3$. To conclude, one may search for other choices of solutions to the considered cubic equation with three unknowns and higher degree diophantine equations with multiple variables.

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