International Research Journal of Education and Technology
Peer Reviewed Journal
ISSN 2581-7795

A SEARCH ON INTEGER SOLUTIONS TO THE NON-HOMOGENEOUS TERNARY QUINTIC EQUATION WITH THREE UNKNOWNS

$$2\left(x^2+y^2\right)-3xy=8z^5$$

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ABSTRACT:

This paper focuses on finding non-zero distinct integer solutions to the Non-Homogeneous Ternary Quintic Diophantine Equation with three unknowns given by $2\left(x^2+y^2\right)-3xy=8z^5$.Various sets of distinct integer solutions to the considered quintic equation are studied through employing the linear transformations $x=u+v, y=u-v\ (u\neq v\neq 0)$and applying the method of factorization.

KEYWORDS: Ternary Quintic, Non-Homogeneous Quintic, Integer Solutions.

INTRODUCTION:

The Theory of Diophantine equations offers a rich variety of fascinating problems. In particular, quintic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity[1-4].For an extensive review of various problems, one may refer [5-16] for quintic equations with three and five unknowns. This communication concerns with yet another interesting non-homogeneous ternary quintic equation with three unknowns represented by $2\left(x^2+y^2\right)-3xy=8z^5$ for determining its infinitely many non-zero integral solutions through different methods.

METHOD OF ANALYSIS

The non- homogeneous quintic Diophantine equation to be solved for its non-zero distinct integer solution is

$$2\left(x^2+y^2\right)-3xy=8z^5 \tag{1}$$

To start with, observe that (1) is satisfied by the following integer triples

$$(x,y,z):\left(2^4\alpha^{5k},2^3\alpha^{5k},2\alpha^{2k}\right),\left(2^4k\left(2k^2-3k+2\right)^2,2^4\left(2k^2-3k+2\right)^2,2\left(2k^2-3k+2\right)\right)$$

However, there are other sets of integer solutions to (1) that are illustrated below:

ILLUSTRATION I:

Introduction of the linear transformations

$$x=u+v,\quad y=u-v,\quad u\neq v\neq 0 \tag{2}$$

in (1) leads to

$$u^2+7v^2=8z^5 \tag{3}$$

The above equation is solved for u, v and z through different methods and using (2), the values of x and y satisfying (1) are obtained which are presented below:

METHOD I:

After performing a few calculations, it is observed that (3) is satisfied by,

$$\begin{aligned} u&=8^3m\left(m^2+7n^2\right)^2\\ v&=8^3n\left(m^2+7n^2\right)^2\\ z&=8\left(m^2+7n^2\right)\end{aligned} \tag{4}$$

In view of (2), the corresponding integer solutions to (1) are found to be

$$x = 8^3(m+n)(m^2+7n^2)^2$$
$$y = 8^3(m-n)(m^2+7n^2)^2 \qquad (5)$$
$$z = 8(m^2+7n^2)$$

METHOD II:

Assume

$$z(a,b) = a^2 + 14b^2 \qquad (6)$$

Case(i)

Write 8 as

$$8 = (1+i\sqrt{7})(1-i\sqrt{7}) \qquad (7)$$

Using (6) and (7) in (3) and employing the method of factorization, consider

$$u + i\sqrt{7}v = (1+i\sqrt{7})(a+i\sqrt{7}b)^5 \qquad (8)$$

Equating the real and imaginary parts, we get

$$u = a^5 - 35a^4b - 70a^3b^2 + 490a^2b^3 + 245ab^4 - 343b^5$$
$$v = a^5 + 5a^4b - 70a^3b^2 - 70a^2b^3 + 245ab^4 + 49b^5 \qquad (9)$$

In view of (2), from (9) we obtain

$$\left.\begin{array}{l} x = 2a^5 - 30a^4b - 140a^3b^2 + 420a^2b^3 + 490ab^4 - 294b^5 \\ y = -40a^4b + 560a^2b^3 - 392b^5 \end{array}\right\} \qquad (10)$$

Thus (6) and (10) represents the integer solutions to (1).

Case(ii)

Write 8 as

$$8 = \frac{(5+i\sqrt{7})(5-i\sqrt{7})}{4} \quad (11)$$

Using (6) and (11) in (3) and employing the method of factorization, consider

$$u + i\sqrt{7}v = (1+i\sqrt{7})\frac{(5+i\sqrt{7})}{2}(a+i\sqrt{7}b)^5 \quad (12)$$

Equating the real and imaginary parts, the values of u and v are obtained as

$$\begin{aligned} u &= \frac{1}{2}[5a^5 - 35a^4b - 350a^3b^2 + 490a^2b^3 + 1225ab^4 - 343b^5] \\ v &= \frac{1}{2}\left[a^5 + 25a^4b - 70a^3b^2 - 350a^2b^3 + 245ab^4 + 245b^5\right] \end{aligned} \quad (13)$$

Replacing a by 2A and b by 2B in (13), we obtain

$$\begin{aligned} u &= 80A^5 - 560A^4B - 5600A^3B^2 + 7840A^2B^3 + 19600AB^4 - 5488B^5 \\ v &= 16A^5 + 400A^4B - 1120A^3B^2 - 5600A^2B^3 + 3920AB^4 + 3920B^5 \end{aligned} \quad (14)$$

and replacing the same prodecure in (6) we get

$$z = 4A^2 + 28B^2 \quad (15)$$

In view of (2), from (14) we obtain

$$\left.\begin{aligned} x &= 96A^5 - 160A^4B - 6720A^3B^2 + 2240A^2B^3 + 23520AB^4 - 1568B^5 \\ y &= 64A^5 - 960A^4B - 4480A^3B^2 + 13440A^2B^3 + 15680AB^4 - 9408B^5 \end{aligned}\right\} \quad (16)$$

Thus (15) and (16) represents the integer solution to (1).

Note 1:

It is seen that 8 is also represented as follows

(i) $8=\frac{(11+i\sqrt{7})(11-i\sqrt{7})}{16}$

(ii) $8=\frac{(31+i\sqrt{7})(31-i\sqrt{7})}{121}$

Following the above procedure as in **METHOD II**, we obtain two more sets of integer solutions to (1) are obtained.

METHOD III

Equation (3) can be written as

$$u^2+7v^2=8z^5*1 \tag{17}$$

Case(i):

Write 1 on the R.H.S. of (17) as

$$1=\frac{(1+3i\sqrt{7})(1-3i\sqrt{7})}{64} \tag{18}$$

Using (6),(7) & (18) in (17) and utilizing the method of factorization, define

$$(u+i\sqrt{7}v)=(1+i\sqrt{7})(a+i\sqrt{7}b)^5\left[\frac{(1+3i\sqrt{7})}{8}\right] \tag{19}$$

Equating the real and imaginary parts, the values of u and v are obtained as

$$\begin{aligned} u &= \frac{1}{2}\left[-5a^5 - 35a^4b + 350a^3b^2 + 490a^2b^3 - 1225ab^4 - 343b^5\right] \\ v &= \frac{1}{2}\left[a^5 - 25a^4b - 70a^3b^2 + 350a^2b^3 + 245ab^4 - 245b^5\right] \end{aligned} \quad (20)$$

Replacing a by 2A and b by 2B in (20), we obtain

$$\begin{aligned} u &= -80A^5 - 560A^4B + 5600A^3B^2 + 7840A^2B^3 - 19600AB^4 - 5488B^5 \\ v &= 16A^5 - 400A^4B - 1120A^3B^2 + 5600A^2B^3 + 3920AB^4 - 3920B^5 \end{aligned} \quad (21)$$

In view of (2), we obtain

$$\left.\begin{aligned} x &= -64A^5 - 960A^4B + 4480A^3B^2 + 13440A^2B^3 - 15680AB^4 - 9408B^5 \\ y &= -96A^5 - 160A^4B + 6720A^3B^2 + 2240A^2B^3 - 23520AB^4 - 1568B^5 \end{aligned}\right\} \quad (22)$$

Thus (15) and (22) represent the integer solution to (1).

Case(ii)

Write 1 on the R.H.S. of (17) as

$$1 = \frac{(3 + i\sqrt{7})(3 - i\sqrt{7})}{16} \quad (23)$$

Using (6),(11) & (23) in (17) and utilizing the method of factorization, define

$$(u + i\sqrt{7}v) = \frac{(5 + i\sqrt{7})}{2}(a + i\sqrt{7}b)^5\left[\frac{(3 + i\sqrt{7})}{4}\right] \quad (24)$$

Equating the real and imaginary parts, the values of u and v are obtained as

$$\begin{aligned} u &= a^5 - 35a^4b - 70a^3b^2 + 490a^2b^3 + 245ab^4 - 343b^5 \\ v &= a^5 + 5a^4b - 70a^3b^2 - 70a^2b^3 + 245ab^4 + 49b^5 \end{aligned} \quad (25)$$

In view of (2), we obtain

$$x = -2a^5 - 30a^4b - 140a^3b^2 + 420a^2b^3 + 490ab^4 - 294b^5$$
$$y = -40a^4b + 560a^2b^3 - 392b^5 \qquad (26)$$

Thus (6) and (26) represent the integer solution to (1).

Note 2:

The integer 1 on the R.H.S of (17) is also expressed as below :

- ✓ $1 = \frac{(3+i4\sqrt{7})(3-i4\sqrt{7})}{121}$
- ✓ $1 = \frac{(9+i5\sqrt{7})(9-5i\sqrt{7})}{256}$
- ✓ $1 = \frac{(7r^2 - s^2 + +i\sqrt{7}2rs)(7r^2 - s^2 - i\sqrt{7}2rs)}{(7r^2+s^2)^2}$

By considering suitable combinations of integers 8 & 1 from Note 1 and Note 2 respectively in (3), some more sets of integer solutions to (1) are obtained.

ILLUSTRATION II:

Introduction of the linear transformations

$$x = 2(u+v), \quad y = 2(u-v), \quad u \neq v \neq 0 \qquad (27)$$

in (1) leads to

$$u^2 + 7v^2 = 2z^5 \qquad (28)$$

The above equation is solved for u, v and z through different methods and using (27), the values of x and y satisfying (1), are obtained which are illustrated below

Method IV:

After some algebra, it is seen that (28) is satisfied by,

$$u = 2^3 m(m^2 + 7n^2)^2$$
$$v = 2^3 n(m^2 + 7n^2)^2$$
$$z = 2(m^2 + 7n^2) \qquad (29)$$

In view of (27), the corresponding integer solutions to (1) are found to be

$$x = 2^4[m^2 + 3n^2]^2(m + n)$$
$$y = 2^4[m^2 + 3n^2]^2(m - n)$$
$$z = 2[m^2 + 3n^2] \qquad (30)$$

CONCLUSION:

In this paper, we have made an attempt to obtain all integer solutions to (1). As (1) is symmetric in x, y, z it is to be noted that, if (x, y, z) is any positive integer solution to (1),then the triples (-x, y, z), (x,-y, z), (x, y,-z), (x, -y, -z), (-x, y, -z), (-x, -y, z) (-x, -y, -z) also satisfy (1).To conclude, one may search for integer solutions to the other choices of non-homogeneous ternary quintic diophantine equations along with suitable properties.

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