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A SEARCH ON THE INTEGER SOLUTIONS TO TERNARY QUADRATIC DIOPHANTINE EQUATION

$$z^2 = 63x^2 + y^2$$

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Abstract:

The homogeneous ternary quadratic diophantine equation given by $\boldsymbol{z^2 = 63x^2 + y^2}$ is analyzed for its non-zero distinct integer solutions through different methods. A few interesting properties between the solutions are presented. Also, formula for generating sequence of integer solutions based on the given solutions are presented.

Keywords: Ternary quadratic, Integer solutions, Homogeneous cone.

Notation:

$$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$$

Introduction:

It is well known that the quadratic diophantine equations with three unknowns(homogenous (or) non-homogenous) are rich in variety [1,2]. In particular, the ternary quadratic diophantine equations of the form $z^2 = Dx^2 + y^2$ are analyzed for values of $D = 29,41,43,47,61,67$ in [3-8]. In this communication, the homogeneous ternary quadratic diophantine equation given by $z^2 = 63x^2 + y^2$ is analyzed for its non-zero distinct integer solutions through different methods. A few interesting properties between the solutions are presented. Also, formulas for generating sequence of integer solutions based on the given solutions are presented.

METHOD OF ANALYSIS

The ternary quadratic diophantine equation to be solved for its integer solutions is

$$\boldsymbol{z^2 = 63x^2 + y^2} \tag{1}$$

We present below different methods of solving (1)

Method:1

(1) is written in the form of ratio as

$$\frac{z+y}{9x} = \frac{7x}{z-y} = \frac{\alpha}{\beta}, \beta \neq 0 \tag{2}$$

which is equivalent to the system of double equations

$$9\alpha x - \beta y - \beta z = 0$$

$$7\beta x + \alpha y - \alpha z = 0$$

Applying the method of cross-multiplication to the above system of equations,one obtains

$$x = x(\alpha,\beta) = 2\alpha\beta$$

$$y = y(\alpha,\beta) = 9\alpha^2 - 7\beta^2$$

$$z = z(\alpha,\beta) = 9\alpha^2 + 7\beta^2$$

which satisfy (1)

Properties:

- $2[y(\alpha,1) + z(\alpha,1)] = 9x^2(\alpha,1)$
- $8[y^3(\alpha,1) + z^3(\alpha,1) + 108\, y(\alpha,1)z(\alpha,1)x^2(\alpha,1) = 729x^6(\alpha,1)$
- $2[y(\alpha,1) + z(\alpha,1)] - 17x(\alpha,1) - t_{72,\alpha}$is a perfect square.
- $y(\alpha,1) - t_{20,\alpha} \equiv 1(mod8)$
- $9y(\alpha,1) - t_{164,\alpha} \equiv 17(mod80)$
- $y(\alpha,1) + z(\alpha,1) - 36t_{3,\alpha} \equiv 4(mod18)$

Note:1

It is observed that (1) may also be represented as below:

$$(i)\frac{z+y}{7x} = \frac{9x}{z-y} = \frac{\alpha}{\beta}, \beta \neq 0$$

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$$(ii)\frac{z+y}{21x}=\frac{3x}{z-y}=\frac{\alpha}{\beta},\beta\neq 0$$

$$(iii)\frac{z+y}{3x}=\frac{21x}{z-y}=\frac{\alpha}{\beta},\beta\neq 0$$

$$(iv)\frac{z+y}{x}=\frac{63x}{z-y}=\frac{\alpha}{\beta},\beta\neq 0$$

Employing the procedure as above in each of the above representations,the corresponding solutions to (1) are presented below:

Solutions for (i):

$$x=2\alpha\beta,\qquad y=7\alpha^2-9\beta^2,\quad z=7\alpha^2+9\beta^2$$

Solutions for (ii):

$$x=2\alpha\beta,\qquad y=21\alpha^2-3\beta^2,\qquad z=21\alpha^2+3\beta^2$$

Solution for (iii):

$$x=2\alpha\beta,\qquad y=3\alpha^2-21\beta^2,\qquad z=3\alpha^2+21\beta^2$$

Solution for (iv):

$$x=2\alpha\beta,\qquad y=\alpha^2-63\beta^2,\quad z=\alpha^2+63\beta^2$$

Method:2

(1)is written as the system of double equations in Table 1 as follows:

Table:1System of Double Equations

System	I	II	III	IV	V	VI	VII	VIII
z+y =	x^2	$3x^2$	$7x^2$	$9x^2$	$21x^2$	$9x$	$21x$	$63x$
z-y =	63	21	9	7	3	$7x$	$3x$	x

Solving each of the above system of double equations,the values of $x, y\& z$ satisfying (1) are obtained .For simplicity and brevity, in what follows, the integer solutions thus obtained are exhibited.

Solutions for system: I

$$x = 2k + 1, \quad y = 2k^2 + 2k - 31, \quad z = 2k^2 + 2k + 32$$

Solutions for system: II

$$x = 2k + 1, \quad y = 6k^2 + 6k - 9, \quad z = 6k^2 + 6k + 12$$

Solutions for system: III

$$x = 2k + 1, \quad y = 14k^2 + 14k - 1, z = 14k^2 + 14k + 8$$

Solutions for system: IV

$$x = 2k + 1, \quad y = 18k^2 + 18k + 1, z = 18k^2 + 18k + 8$$

Solutions for system: V

$$x = 2k + 1, \quad y = 42k^2 + 42k + 9, z = 42k^2 + 42k + 12$$

Solutions for system: VI

$$x = k, \quad y = k, \quad z = 8k$$

Solutions for system: VII

$$x = k, \quad y = 9k, \quad z = 12k$$

Solutions for system: VIII

$$x = k, \quad y = 31k, \quad z = 32k$$

Method:3

Let $z = y + k, \ k \neq 0$ (3)

$$\therefore (1) \Rightarrow 2ky = 63x^2 - k^2$$

Assume

$$x = k(2\alpha + 1) \tag{4}$$

$$\therefore y = 63(2k\alpha^2 + 2k\alpha) + 31k \tag{5}$$

In view of (3),

$$z = 63(2k\alpha^2 + 2k\alpha) + 32k \qquad (6)$$

Note that (4), (5), (6)satisfy (1).

Method:4

(1) is written as

$$y^2 + 63x^2 = z^2 = z^2 * 1 \qquad (7)$$

Assume z as

$$z = a^2 + 63b^2 \qquad (8)$$

Write 1 as

$$1 = \frac{(1 + i\sqrt{63})(1 - i\sqrt{63})}{64} \qquad (9)$$

Using (8)& (9) in (7) and employing the method of factorization,consider

$$(y + i\sqrt{63}\,x) = (a + i\sqrt{63}\,b)^2 . \frac{(1 + i\sqrt{63})}{8}$$

Equating the real&imaginary parts, it is seen that

$$\left.\begin{aligned} x &= \frac{1}{8}[a^2 - 63b^2 + 2ab] \\ y &= \frac{1}{8}[(a^2 - 63b^2) - 126ab] \end{aligned}\right\} \qquad (10)$$

Since our interest is to find the integer solutions,replacing $a \; by \; 8A \& b \; by \; 8B$ in (10)& (8),the corresponding integer solutions to (1) are given by

$$x = x(A, B) = 8[A^2 - 63B^2 + 2AB],$$

$$y = y(A, B) = 8[A^2 - 63B^2 - 126AB],$$

$$z = z(A, B) = 64[A^2 + 63B^2]$$

Properties:

- $x(A, A+1) - y(A, A+1) = 2048t_{3,A}$
- $[x(\alpha^2, \alpha+1) - y(\alpha^2, \alpha+1) - 64t_{34,\alpha^2}]$ is 31 times a square.
- $8y(A, A+1) + z(A, A+1) = 64[t_{6,A} - 252t_{3,A}]$
- $8x(A, A+1) + z(A, A+1) = 64[t_{6,A} + 4t_{3,A}]$
- $4(x(A, A+1) + y(A, A+1)) + z(A, A+1) = 64[t_{6,A} - 124t_{3,A}]$

Note:2

In addition to (9),1 may also be represented as follows:

$$(v) \qquad 1=\frac{(9+i\sqrt{63})(9-i\sqrt{63})}{144}$$

$$(vi) \qquad 1=\frac{\left[(63\alpha^2-\beta^2)+i\sqrt{63}\,.2\alpha\beta\right]\left[(63\alpha^2-\beta^2)-i\sqrt{63}\,.2\alpha\beta\right]}{(63\alpha^2+\beta^2)^2}$$

For the above choices,the corresponding values of x, y, z satisfying (1) are given below:

Solutions for $(\boldsymbol{v})$:

$$x=12[A^2-63B^2+18AB],$$

$$y=12[9A^2-567AB-126AB],$$

$$z=144[A^2+63B^2]$$

Solutions for $(\boldsymbol{vi})$:

$$x=(63\alpha^2+\beta^2)[2\alpha\beta(A^2-63B^2)+2AB(63\alpha^2-\beta^2)],$$

$$y=(63\alpha^2+\beta^2)[(63\alpha^2-\beta^2)(A^2-63B^2)-252\alpha\beta AB],$$

$$z=(63\alpha^2+\beta^2)^2(A^2+63B^2)$$

GENERATION OF SOLUTIONS

Different formulas for generating sequence of integer solutions based on the given solution are presented below:

Let (x_0, y_0, z_0) be any given solution to (1)

Formula: 1

Let (x_1, y_1, z_1)given by

$$x_1=x_0+h,\ y_1=y_0,\ z_1=-z_0+8h \tag{11}$$

be the 2nd solution to (1).Using (11) in (1) and simplifying, one obtains

$$h=126x_0+16z_0$$

In view of (11), the values of x_1 and z_1 is written in the matrix form as

$$(x_1, z_1)^t = M(x_0, z_0)^t$$

where

$$M = \begin{pmatrix} 127 & 16 \\ 1008 & 127 \end{pmatrix} \text{ and } t \text{ is the transpose}$$

The repetition of the above process leads to the n^{th} solutions x_n, z_n given by

$(x_n, z_n)^t = M^n(x_0, z_0)^t$

If α, β are the distinct eigenvalues of M, then

$\alpha = 127 + 48\sqrt{7}, \beta = 127 - 48\sqrt{7}$

We know that

$$M^n = \frac{\alpha^n}{(\alpha - \beta)}(M - \beta I) + \frac{\beta^n}{(\beta - \alpha)}(M - \alpha I), I = 2 \times 2 \; identity\; matrix$$

Thus, the general formulas for integer solutions to (1) are given by

$$x_n = \left(\frac{\alpha^n + \beta^n}{2}\right)x_0 + \left(\frac{\alpha^n - \beta^n}{6\sqrt{7}}\right)z_0$$

$$y_n = y_0$$

$$z_n = \frac{21}{2\sqrt{7}}(\alpha^n - \beta^n)x_0 + \left(\frac{\alpha^n + \beta^n}{2}\right)z_0$$

Formula: 2

Let (x_1, y_1, z_1)given by

$$x_1 = 3x_0,\ y_1 = 3y_0 + h,\ z_1 = 2h - 3z_0 \qquad (12)$$

be the 2nd solution to (1).Using (12) in (1) and simplifying, one obtains

$$h = 2x_0 + 16z_0$$

In view of (12), the values of y_1 and z_1 is written in the matrix form as

$$(y_1, z_1)^t = M(y_0, z_0)^t$$

where

$$M = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \text{ and } t \text{ is the transpose}$$

The repetition of the above process leads to the n^{th} solutions y_n, z_n given by

$$(y_n, z_n)^t = M^n (y_0, z_0)^t$$

If α, β are the distinct eigenvalues of M, then

$$\alpha = 1, \beta = 9$$

Thus, the general formulas for integer solutions to (1) are given by

$$x_n = 3^n x_0$$

$$y_n = \left(\frac{9^n + 1}{2}\right) y_0 + \left(\frac{9^n - 1}{2}\right) z_0$$

$$z_n = \left(\frac{9^n - 1}{2}\right) y_0 + \left(\frac{9^n + 1}{2}\right) z_0$$

Formula: 3

Let (x_1, y_1, z_1)given by

$$x_1 = -64x_0 + h,\ y_1 = -64y_0 + h,\ z_1 = 64z_0 \qquad (13)$$

be the 2nd solution to (1).Using (13) in (1) and simplifying, one obtains

$$h = 126x_0 + 2y_0$$

In view of (13), the values of x_1 and y_1 is written in the matrix form as

$$(x_1, y_1)^t = M(x_0, y_0)^t$$

where

$$M = \begin{pmatrix} 62 & 2 \\ 126 & -62 \end{pmatrix} \text{ and } t \text{ is the transpose}$$

The repetition of the above process leads to the n^{th} solutions x_n, y_n given by

$$(x_n, y_n)^t = M^n (x_0, y_0)^t$$

If α, β are the distinct eigenvalues of$M$, then

$$\alpha = 64,\ \beta = -64$$

Thus, the general formulas for integer solutions to (1) are given by

$$x_n = 64^{n-1}(63 + (-1)^n)x_0 + 64^{n-1}(1 - (-1)^n)y_0$$

$$y_n = 63.64^{n-1}(1 - (-1)^n)x_0 + 64^{n-1}(1 + 63(-1)^n)y_0$$

$$z_n = 64^n z_0$$

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