International Research Journal of Education and Technology
ISSN 2581-7795

On the homogeneous quadratic Diophantine equation with three unknowns

$$7x^2 + y^2 = 448z^2$$

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Abstract : The ternary quadratic equation given by $4x^2 - 12xy + 21y^2 = 13z^2$ isconsidered and searched for its many different integer solution . Five different choices of integer solution of the above equations are presented .A few interesting relations between the solutions and special polygonal numbers are presented.

Key words: ternary quadratic, integer solutions

MSC subject classification :11D09

1.INTRODUCTION:

The Diophantine equation offer an unlimited field for research due to their variety[1-3].In particular ,one may refer [4-15] for quadratic equations with three unknowns . This communication concerns with yet another interesting equation $4x^2 - 12xy + 21y^2 = 13z^2$ representing homogeneous equation with three for determining its infinitely Many non -zero integral points. Also ,few interesting relations among the solutions are presented.

2.NOTATIONS:

- $t_{m,n} = n^{th}$ term of a regular polygon with m sides.

$$= n\left(1+\frac{(n-1)(m-2)}{2}\right)$$

- PR_n =pronic number of rank n

$$= n(n+1)$$

3.METHOD OF ANALYSIS

The quadratic diophantine equations with three unknowns to be solved is given by,

$$7x^2 + y^2 = 448z^2 \quad (1)$$

Substituting,

$$\left.\begin{aligned} x &= X - T \\ y &= X + 7T \end{aligned}\right\} \qquad (2)$$

in (3.1),we get,

$$7(x-T)^2 + (X+7T)^2 = 448z^2$$

Then,

$$X^2 + 7T^2 = 56z^2 \qquad (3)$$

(3) is solved through different approaches and the different patterns of solution of (1) obtained are presented below,

PATTERN 1

Assume ,

$$z = a^2 + 7b^2 \qquad (4)$$

Write 56 as,

$$56 = \left(7 + i\sqrt{7}\right)\left(7 - i\sqrt{7}\right)$$

(3) can be written as,

$$\left(X + i\sqrt{7}T\right)\left(X - i\sqrt{7}T\right) = \left(7 + i\sqrt{7}\right)\left(7 - i\sqrt{7}\right)\left(a + i\sqrt{7}b\right)^2\left(a - i\sqrt{7}b\right)^2$$

Equating positive terms

$$\left(X + i\sqrt{7}T\right) = \left(7 + i\sqrt{7}\right)\left(a + i\sqrt{7}b\right)^2$$

Equating real and imaginary parts

$$X = 7a^2 - 49b^2 - 14ab$$

$$T = a^2 - 7b^2 + 14ab$$

From (2),

$$x = X - T$$

$$y = X + 7T$$

We obtain the non-zero distinct integral solution of (1) as

$$x(a,b) = 6a^2 - 42b^2 - 28ab$$

$$y(a,b) = 7a^2 - 98b^2 + 84ab$$

$$z(a,b) = a^2 + 7b^2$$

PROPERTIES:

$$x(a,1) + y(a,1) + 43t_{4,a} - 56\,\mathrm{Pr}_a = -140$$

$$x(a,1) - y(a,1) - 111t_{4,a} + 112\,\mathrm{Pr}_a = 56$$

$$x(a,1) + z(a,1) - 35t_{4,a} + 28\,\mathrm{Pr}_a = -35$$

PATTERN 2

Assume ,

$$z = a^2 + 7b^2$$

Write 56 as,

$$56 = \frac{\left(7 + 5i\sqrt{7}\right)\left(7 - 5i\sqrt{7}\right)}{7^2}$$

(3) can be written as,

$$(X + i\sqrt{7}T)\left(X - i\sqrt{7}T\right) = \frac{\left(7 + 5i\sqrt{7}T\right)\left(7 - 5i\sqrt{7}T\right)}{2^2}\left(a + i\sqrt{7}b\right)^2\left(a - i\sqrt{7}b\right)^2$$

Taking positive terms,

$$(X + i\sqrt{7}T) = \frac{1}{2}(7 + 5i\sqrt{7})(a + i\sqrt{7}b)^2$$

$$(X + i\sqrt{7}T) = \frac{1}{2}(7a^2 - 49b^2 - 70ab) + i\sqrt{7}(5a^2 - 35b^2 + 14ab)$$

Equating real and imaginary parts,

$$X = \frac{1}{2}(7a^2 - 49b^2 - 70ab)$$

$$T = \frac{1}{2}(5a^2 - 35b^2 + 14ab)$$

From (2)

$$x = X - T$$

$$y = X + 7T$$

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Assume a=2A, b=2B ,We obtain the non-zero distinct integral solution of (1) as

$$x(A,B)=4A^2-28B^2-168AB$$

$$y(A,B)=84A^2-588B^2+56AB$$

$$z(A,B)=4A^2+28B^2$$

PROPERTIES:

$$x(A,1)+y(A,1)+200t_{4,A}+112\Pr_A=-616$$

$$x(A,1)-y(A,1)-144t_{4,A}+224\Pr_A=560$$

$$x(A,1)+z(A,1)-176t_{4,A}+168\Pr_A=0$$

PATTERN 3

Assume ,

$$z=a^2+7b^2$$

56 can be written as,

$$56=\frac{\left(X+11i\sqrt{7}\right)\left(X-11i\sqrt{7}\right)}{4^2}$$

(3) can be write as,

$$(X+i\sqrt{7}T)(X-i\sqrt{7}T)=\frac{(7+i11\sqrt{7})(7+11i\sqrt{7})}{4^2}(a+i\sqrt{7}b)^2(a-i\sqrt{7}b)^2$$

Taking positive terms,

$$(X+i\sqrt{7}T)=\frac{7+11i\sqrt{7}}{4}(a+i\sqrt{7}b)^2$$

$$=\frac{1}{4}\left[\left(7a^2-49b^2-154ab\right)+i\sqrt{7}(11a^2-77b^2+14ab)\right]$$

Equating real and imaginary parts,

$$X=\frac{1}{4}\left(7a^2-49b^2-154ab\right)$$

$$T=\frac{1}{4}\left(11a^2-77b^2+14ab\right)$$

Assume $a = 4A, b = 4B$, we obtain the non-zero distinct integral soln of (1)

$$x(A,B) = -16A^2 + 112B^2 - 672AB$$

$$y(A,B) = 336A^2 - 2352B^2 - 224AB$$

$$z(A,B) = 16A^2 + 112B^2$$

PROPERTIES

$$x(A,1) + y(A,1) - 1216t_{4,A} + 896\,\mathrm{Pr}_A = -2240$$

$$x(A,1) - y(A,1) - 96t_{4,A} + 448\,\mathrm{Pr}_A = 2464$$

$$x(A,1) + z(A,1) - 672t_{4,A} + 672\,\mathrm{Pr}_A = 224$$

PATTERN 4

Assume ,

$$z = a^2 + 7b^2$$

56 can also be written as

$$56 = (7 + i\sqrt{7})(7 - i\sqrt{7})$$

1 can be write as ,

$$1 = \frac{(3 + i\sqrt{7})(3 - i\sqrt{7})}{4^2}$$

Substitute this in (4),we get,

$$(X + i\sqrt{7}T)(X - i\sqrt{7}T) = (7 + i\sqrt{7})(7 + i\sqrt{7})\frac{(3 + i\sqrt{7})(3 - i\sqrt{7})}{4^2}(a + i\sqrt{7}b)^2(a - i\sqrt{7}b)^2$$

Taking positive terms,

$$(X + i\sqrt{7}T) = (7 + i\sqrt{7})\frac{(3 + i\sqrt{7})}{4}(a + i\sqrt{7}b)^2$$

$$= \frac{1}{4}\left[(14a^2 - 98b^2 - 140ab) + i\sqrt{7}(10a^2 - 70b^2 + 28ab)\right]$$

Equating real and imaginary parts,

$$X = \frac{1}{4}(14a^2 - 98b^2 - 140ab)$$

$$T = \frac{1}{4}(10a^2 - 70b^2 + 28ab)$$

From (2)

$x = X - T$

$y = X + 7T$

Assume a=4A, b=4B, We obtain the non-zero distinct integral solution of (1)as,

$$x(A,B) = 16A^2 - 112B^2 - 672AB$$

$$y(A,B) = 336A^2 - 2352B^2 + 224AB$$

$$z(A,B) = 16A^2 + 112B^2$$

PROPERTIES:

$$x(A,1) + y(A,1) - 800t_{4,A} + 448\Pr_A = -2464$$

$$x(A,1) - y(A,1) - 576t_{4,A} + 896\Pr_A = 2240$$

$$x(A,1) + z(A,1) - 704t_{4,A} + 672\Pr_A = 0$$

PATTERN 5

$$X^2 + 7T^2 = 56z^2$$

$$\Rightarrow X^2 + 7T^2 = 7z^2 + 49z^2$$

$$\Rightarrow X^2 - 49z^2 = 7(z^2 - T^2)$$

$$\Rightarrow (X + 7z)(X - 7z) = 7(z + T)(z - T) \quad (5)$$

Case 1

(5) can be written as ,

$$\frac{X + 7z}{7(z + T)} = \frac{z - T}{X - 7z} = \frac{\alpha}{\beta}$$

Which is equivalent to the system of double equation as,

$$\left.\begin{array}{l} X\beta - 7\alpha T + z(7\beta - 7\alpha) = 0 \\ \alpha X - \beta T - z(7\alpha + \beta) = 0 \end{array}\right\} \quad (6)$$

Solving (6) by the method of cross multiplication, we get

$$\left.\begin{array}{l} X = -49\alpha^2 - 7\beta^2 \\ T = -7\alpha^2 - \beta^2 \\ z = 7\alpha^2 + \beta^2 \end{array}\right\} \quad (7)$$

Substiuting (7) in(4), the non-zero distinct integer solution of (1) are given by,

$$x(\alpha,\beta) = -42\alpha^2 - 6\beta^2$$
$$y(\alpha,\beta) = -98\alpha^2 - 14\beta^2$$
$$z(\alpha,\beta) = 7\alpha^2 + \beta^2$$

PROPERTIES:

$$x(\alpha,1) + y(\alpha,1) + 140t_{4,\alpha} = -20$$
$$x(\alpha,1) + y(\alpha,1) - 56t_{4,\alpha} \equiv 0(\bmod 2)$$
$$x(\alpha,1) + y(\alpha,1) + 35t_{4,\alpha} = -5$$

Case 2

$$\frac{X+7z}{z-T} = \frac{7(z+T)}{X-7z} = \frac{\alpha}{\beta}$$

Which is equivalent to the system of double equation as,

$$\left.\begin{aligned} -\beta X - \alpha T + z(\alpha - 7\beta) = 0 \\ \alpha X - 7\beta T - 7z(\alpha+\beta) = 0 \end{aligned}\right\} \quad (8)$$

Solving (8) by the method of cross multiplication, we get

$$\left.\begin{aligned} X &= 7\alpha^2 + 14\alpha\beta - 49\beta^2 \\ T &= \alpha^2 - 14\alpha\beta - 7\beta^2 \\ z &= \alpha^2 + 7\beta^2 \end{aligned}\right\} \quad (9)$$

Substituting (9) in (2) ,the non-zero distinct integer solution of (1) are given by,

$$x(\alpha,\beta) = 6\alpha^2 - 42\beta^2 + 28\alpha\beta$$
$$y(\alpha,\beta) = 14\alpha^2 - 98\beta^2 - 84\alpha\beta$$
$$z(\alpha,\beta) = \alpha^2 + 7\beta^2$$

PROPERTIES

$$x(\alpha,1) + y(\alpha,1) - 764t_{4,\alpha} + 54\,\mathrm{Pr}_\alpha = -140$$
$$x(\alpha,1) - y(\alpha,1) + 120t_{4,\alpha} - 112\,\mathrm{Pr}_\alpha \equiv 0(\bmod 2)$$

$$x(\alpha,1)+z(\alpha,1)+21t_{4,\alpha}-28\Pr_{\alpha}=-35$$

Case 3

$$\frac{X-7z}{z+7}=\frac{7(z-T)}{X+7z}=\frac{\alpha}{\beta}$$

Which is equivalent to the system of double equation as,

$$\left.\begin{aligned}\beta X-z(7\beta+\alpha)-\alpha T=0\\ -\alpha X+z(7\beta-7\alpha)-7\beta T=0\end{aligned}\right\} \qquad (10)$$

Solving (3.10) by the method of cross multiplication, we get

$$\left.\begin{aligned}X&=-7\alpha^2+14\alpha\beta+49\beta^2\\ z&=\alpha^2+7\beta^2\\ T&=-\alpha^2-14\alpha\beta+7\beta^2\end{aligned}\right\} \qquad (11)$$

Substituting (11) in (2) ,the non-zero distinct integer solution of (1) are given by,

$$x(\alpha,\beta)=-6\alpha^2+28\alpha\beta+42\beta^2$$

$$y(\alpha,\beta)=-14\alpha^2-84\alpha\beta+98\beta^2$$

$$z(\alpha,\beta)=\alpha^2+7\beta^2$$

PROPERTIES:

$$x(\alpha,1)+y(\alpha,1)-36t_{4,\alpha}+56\Pr_{\alpha}=140$$

$$x(\alpha,1)-y(\alpha,1)+104t_{4,\alpha}-112\Pr_{\alpha}\equiv-56$$

$$x(\alpha,1)+z(\alpha,1)+33t_{4,\alpha}-28\Pr_{\alpha}\equiv 0(\bmod 7)$$

Case 4

$$\frac{X+7z}{z+T}=\frac{7(z-T)}{X-7z}=\frac{\alpha}{\beta}$$

Which is equivalent to the system of double equation as,

$$\left.\begin{aligned}\beta X+z(7\beta-\alpha)-\alpha T=0\\ -\alpha X+z(7\beta+7\alpha)-7\beta T=0\end{aligned}\right\} \qquad (12)$$

Solving (3.12) by the method of cross multiplication, we get

$$\left.\begin{aligned} X &= 7\alpha^2 + 14\alpha\beta - 49\beta^2 \\ z &= \alpha^2 + 7\beta^2 \\ T &= -\alpha^2 + 14\alpha\beta + 7\beta^2 \end{aligned}\right\} \quad (13)$$

Substituting (13) in (2) ,the non-zero distinct integer solution of (1) are given by,

$$x(\alpha,\beta) = 8\alpha^2 - 56\beta^2$$
$$y(\alpha,\beta) = 112\alpha\beta$$
$$z(\alpha,\beta) = \alpha^2 + 7\beta^2$$

PROPERTIES:

$$x(\alpha,1) + y(\alpha,1) + 104t_{4,\alpha} - 112\,\mathrm{Pr}_\alpha = -56$$
$$x(\alpha,1) - y(\alpha,1) - 120t_{4,\alpha} + 112\,\mathrm{Pr}_\alpha = -56$$
$$y(\alpha,1) + z(\alpha,1) + 111t_{4,\alpha} - 112\,\mathrm{Pr}_\alpha \equiv 0 \pmod 7$$

CONCLUSION:

In this paper,we have presented infinitely many non-zero distinct integer solution to the ternary quadratic equation $7x^2+y^2=448z^2$representing homogeneous cone . As diophantine equation are rich in variety , to conclude, one may search for other forms of three dimentional surfaces , namely , non-homogeneous cone , paraboloid , ellipsoid ,hyperbolic paraboloid and so on for finding integral points on them and corresponding properties .

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