



A Deviant Pattern of Integer Triples

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Abstract

This paper exhibits formula for forming a sequence of non-zero distinct integer triples (x, y, z) such that their squares form a harmonic progression.

Keywords : Integer triple ,Harmonic progression

Introduction

In mathematics , integers are plenty.A pattern in integers is a repeating arrangement of integers. That is , it is an ordered set of integers. Patterns in integers have a deep rooted history. Recognizing integer patterns is a significant problem solving skill. The integer patterns have attracted and motivated many mathematicians since antiquity[1-7]. In [8] ,the authors have shown that there exists an integer N such that the three digit integer $abc = N + 2$ and its reverse

integer $cba = 2N$, where a, b, c are three non-zero single digit integers. In [9], the authors show an integer pattern given by $(ab)^2 + (cd)^2 + (ef)^2 = (ba)^2 + (dc)^2 + (fe)^2$, where a, b, c, d, e and f be six non-zero single digit integers respectively.

In this paper, formula for forming a sequence of non-zero distinct integer triples (x, y, z) such that their squares form a harmonic progression is exhibited.

Method of analysis

Let x, y, z be any three non-zero distinct integers such that their squares x^2, y^2, z^2 form a harmonic progression. That is, $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ form an arithmetic progression and represented mathematically as

$$\frac{2}{y^2} = \frac{1}{x^2} + \frac{1}{z^2}$$

$$\Rightarrow 2x^2z^2 = y^2(x^2 + z^2) \quad (1)$$

The substitution of the transformations

$$x = u + kv, z = u - kv, u \neq kv \neq 0 \quad (2)$$

in (1) leads to

$$u^4 - (2k^2v^2 + y^2)u^2 + k^4v^4 - k^2v^2y^2 = 0 \quad (3)$$

Treating (3) as a quadratic in u^2 and solving for the same, one obtains

$$u^2 = \frac{1}{2}[2k^2v^2 + y^2 \pm y\sqrt{y^2 + 8k^2v^2}] \quad (4)$$

The square-root on the R.H.S. of (4) is removed when

$$v = 2rs, y = 8k^2r^2 - s^2 \quad (5)$$

Choice 1

Substituting (5) in (4) and taking the positive sign before the square-root, we have

$$u^2 = 64k^4r^4 - 4k^2r^2s^2 \quad (6)$$

Again, taking



$$u = 8k^2rU, s = 4kS \quad (7)$$

in (6), one obtains

$$r^2 = U^2 + S^2 \quad (8)$$

which is the well-known Pythagorean equation satisfied by

$$U = 2mn, S = m^2 - n^2, r = m^2 + n^2 \quad (9)$$

Using (9) in (7) and (5), one has

$$\begin{aligned} u &= 16k^2mn(m^2 + n^2) \\ s &= 4k(m^2 - n^2) \\ v &= 8k(m^4 - n^4) \end{aligned} \quad (10)$$

and

$$y = y(m, n) = 8k^2(m^2 + n^2)^2 - 16k^2(m^2 - n^2)^2 \quad (11)$$

From (10) and (2), we have

$$\begin{aligned} x &= x(m, n) = 8k^2(m^2 + n^2)(2mn + m^2 - n^2) \\ z &= z(m, n) = 8k^2(m^2 + n^2)(2mn - m^2 + n^2) \end{aligned} \quad (12)$$

Thus, (11) and (12) satisfy (1).

Remark 1

Apart from (9), the integer solutions to (8) may be taken as

$$U = m^2 - n^2, S = 2mn, r = m^2 + n^2 \quad (13)$$

Using (13) in (7) and (5), one has

$$\begin{aligned} u &= 8k^2(m^4 - n^4) \\ s &= 8kmn \\ v &= 16kmn(m^2 + n^2) \end{aligned} \quad (14)$$

and

$$y = y(m, n) = 8k^2(m^2 + n^2)^2 - 64k^2m^2n^2 \quad (15)$$

From (14) and (2), we have

$$\begin{aligned} x &= x(m, n) = 8k^2(m^2 + n^2)[m^2 - n^2 + 2mn] \\ z &= z(m, n) = 8k^2(m^2 + n^2)[m^2 - n^2 - 2mn] \end{aligned} \quad (16)$$

Thus, (15) and (16) satisfy (1).

Choice 2

Substituting (5) in (4) and taking the negative sign before the square-root ,
we have

$$u^2 = s^4 - 4k^2 r^2 s^2 = s^2 (s^2 - 4k^2 r^2) \quad (17)$$

Again, taking

$$u = s U \quad (18)$$

in (17), it is written as

$$s^2 = U^2 + (2kr)^2 \quad (19)$$

which is in the form of Pythagorean equation satisfied by

$$U = k^2 m^2 - n^2, s = k^2 m^2 + n^2, r = mn \quad (20)$$

Using (20) in (18) and (5), we have

$$u = s(k^2 m^2 - n^2) \quad (21)$$

$$v = 2mn(k^2 m^2 + n^2)$$

and

$$y = y(m, n) = 8k^2 m^2 n^2 - (k^2 m^2 + n^2)^2 \quad (22)$$

From (21) and (2), we have

$$x = x(m, n) = (k^2 m^2 + n^2)[k^2 m^2 - n^2 + 2kmn] \quad (23)$$

$$z = z(m, n) = (k^2 m^2 + n^2)[k^2 m^2 - n^2 - 2kmn]$$

Thus, (22) and (23) satisfy (1).

Remark 2

Apart from (20), the solutions to (19) can be taken as

$$U = 8k^2 mn, r = 2k(m^2 - n^2), s = 4k^2(m^2 + n^2) \quad (24)$$

Using (24) in (18) and (5), we get

$$u = 32k^4 mn(m^2 + n^2) \quad (25)$$

$$v = 16k^3(m^4 - n^4)$$

and

$$y = y(m, n) = 32k^4(m^2 - n^2)^2 - 16k^4(m^2 + n^2)^2 \quad (26)$$

From (25) and (2) , we have

$$\begin{aligned} x &= x(m, n) = 16k^4 (m^2 + n^2) [2mn + m^2 - n^2] \\ z &= z(m, n) = 16k^4 (m^2 + n^2) [2mn - m^2 + n^2] \end{aligned} \quad (27)$$

Thus, (26) and (27) satisfy (1).

Conclusion

As it is well-known that the integers are rich in variety , the readers and researchers in this area of number patterns may attempt to obtain new and different choices of integer patterns.

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